enumeration algorithms

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zoo of problems

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- Counting problem: compute $|A(X)|$
- Enumeration problem: Output $A(X)$

example

A clique C of a graph is said to be maximal if it is not included in any other clique.

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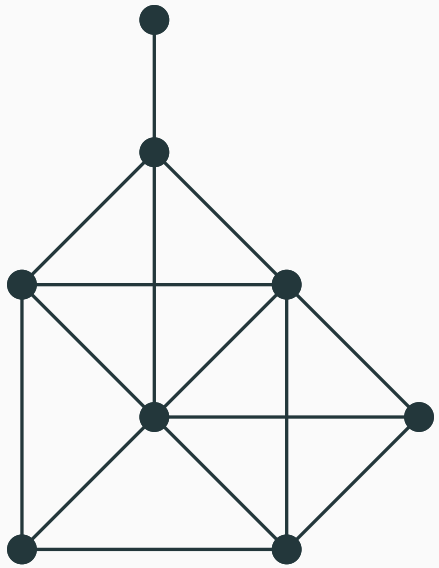
Problem: Maximal Cliques Enumeration

Input: A graph G

Output: All maximal cliques of G

example



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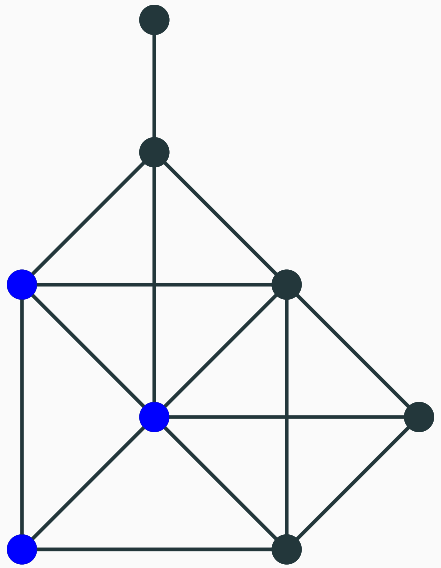

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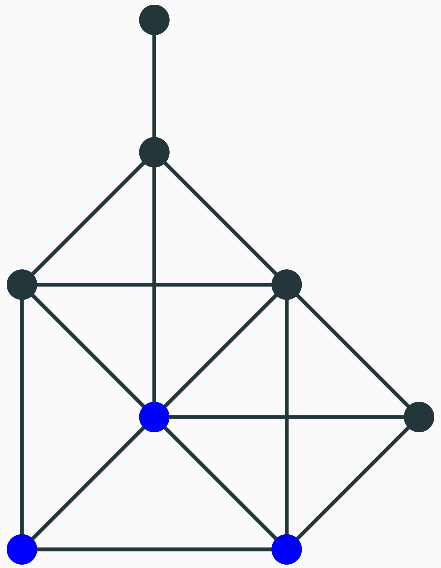

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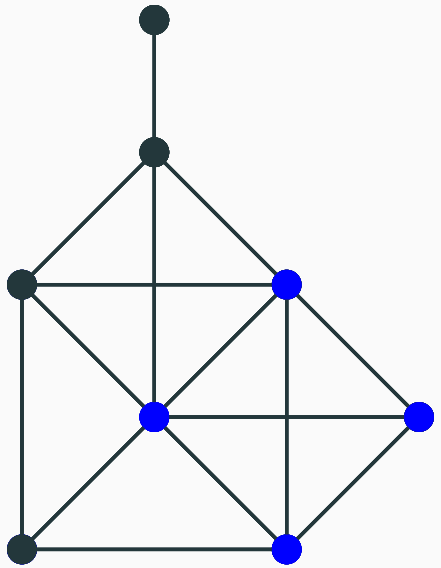

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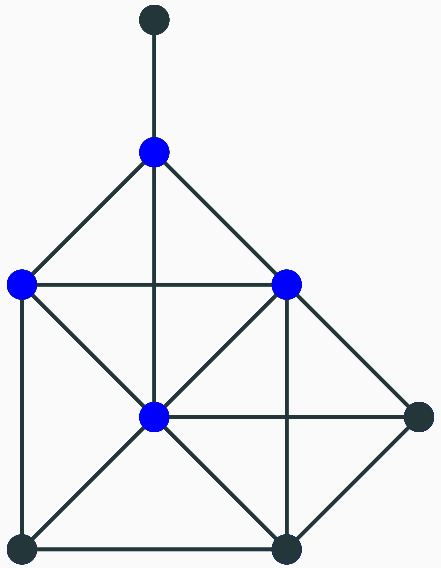

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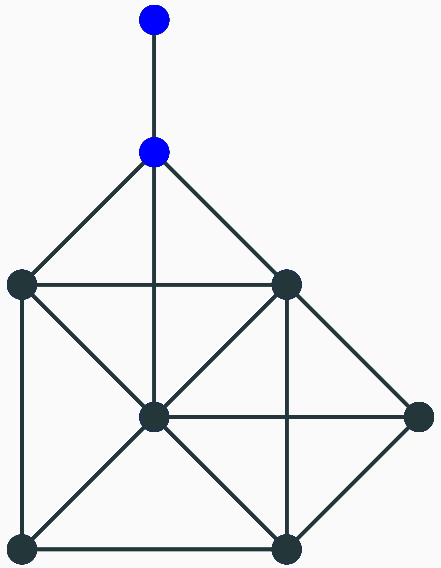

complexity

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- The number of outputs of an enumeration problem may be exponential in the size of the input.
- For instance: the number of maximal cliques of a graph can be exponential on n (a graph can have $3^{n/3}$ maximal cliques)
- We cannot hope to have a algorithm whose running time is polynomial in the input (you need at least one operation per output).

dilemma

- What is an efficient enumeration algorithm?

dilemma

- What is an efficient enumeration algorithm?
- How to define tractability for enumeration problems?

dilemma

- What is an efficient enumeration algorithm?
- How to define tractability for enumeration problems?
- How to measure the complexity of an enumeration algorithm.

two schools

two schools

Two points of view

Input sensitive

- The complexity is measured in term of the input size only.

Output sensitive

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Output sensitive

- The complexity is measured in term of the input size and the output size.
- We then can talk about "polynomial" algorithm even if the number of solutions can be exponentially large in the input size.

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Input sensitive

- The goal is to find an exponential algorithm with the smallest complexity

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Input sensitive

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- Typical example is to find an algorithm of complexity α^n with alpha as small as possible (usually smaller than 2)

Output sensitive

- We try to find an algorithm polynomial in the input size and output size
- ideally we want a complexity linear in the output size.

Exemple

Input sensitive

Given a graph G with n vertices, there exists an algorithm that enumerates all maximal cliques in $O(3^{\frac{n}{3}}) \sim O(1.442^n)$.

Output sensitive

Exemple

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Given a graph G with n vertices, there exists an algorithm that enumerates all maximal cliques in $O(3^{\frac{n}{3}}) \sim O(1.442^n)$.

Output sensitive

Given a graph G with n vertices, there exists an algorithm that enumerates all maximal cliques in $O(n^3|\mathcal{C}|)$ where $|\mathcal{C}|$ denotes the number of maximal cliques of G.

Input sensitive

Pros

- We know in advance the running time of the algorithm.

Cons

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- The complexity of the algorithm gives a combinatorial bound on the number of possible solutions

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Cons

- We don't know the relative efficiency of the algorithm. Is it far to be optimal or not?
- The complexity measure depends on the number of solutions that one instance can have.
- We don't know the behavior of the algorithm on instances with few solutions.

output sensitive

Output sensitive

Pros	Cons
• The efficiency does not depend on the problem.	

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- We can see if an algorithm is far from the (theoretical) optimal complexity.
- We can classify enumeration problems by their output sensitive complexity.

Cons

- We don't know in advance how many times the algorithm will run.
- Doesn't give any information on the maximal number of solutions an instance can have.

input sensitive enumeration

input sensitive algorithm

Comes from exact exponential algorithm community

Methods used:

- Branch and reduce algorithm
- Measure and conquer algorithm
- Classical dynamic programming algorithm.

branch and reduce algorithm

A branch and reduce algorithm is a recursive algorithm that divides the instance in some smaller instances based on a set of rules. There are two kinds of rules:

- **Reduction rules:** They just reduce the size of the instance. They are usually based on easy observations.
- **Branching rules:** The rules that effectively divide the problem in smaller sub-problems.

toy example

Enumeration of maximal independent sets of a graph G of maximum degree 2

- S: The current set of vertices taken in the solution.
- F: The set of "free vertices"; the vertices that are allowed to be taken in the solution.

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Reduction Rules:

- If F contains a degree 0 vertex, then add it to S and remove it from F

Enumeration of maximal independent sets of a graph G of maximum degree 2

Branching Rules:

- If F contains a degree 1 vertex u with neighbour v, then branch in the following ways:
  - Add u to S and remove u and v from F.
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Enumeration of maximal independent sets of a graph G of maximum degree 2

Branching Rules:

- If F contains a degree 2 vertex u with neighbour v, w, then branch in the following ways:
  - Add u to S and remove u, v and w from F.
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complexity analysis

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- To determine the complexity of the previous algorithm, let us analyse its recursion tree.
- We want to determine the number of leaves of the tree.
- In each internal node of the tree we count the number of sub-problems we create and how the size of these sub-problems decreases.

branching vectors

Branching Rules:

- If F contains a degree 1 vertex u with neighbour v, then branch in the following ways:
  - Add u to S and remove u and v from F. $|F|$ decreases by 2
  - Add v to S and remove u, v and all neighbours of v from F. $|F|$ decreases by 2

We branch in two sub-problems each whose sizes decrease by 2.

It is described by the Branching vector $(2, 2)$

branching vectors

Branching Rules:

- If F contains a degree 2 vertex u with neighbour v, w, then branch in the following ways:
  - Add u to S and remove u, v and w from F. $|F|$ decreases by 3
  - Add v to S and remove u, v and all neighbours of v from F. $|F|$ decreases by 3
  - Add w to S and remove u, w and all neighbours of w from F. $|F|$ decreases by 3

We branch in three sub-problems each whose sizes decrease by 3.

Branching vector: $(3, 3, 3)$

example of tree with branching vector $(3, 3)$

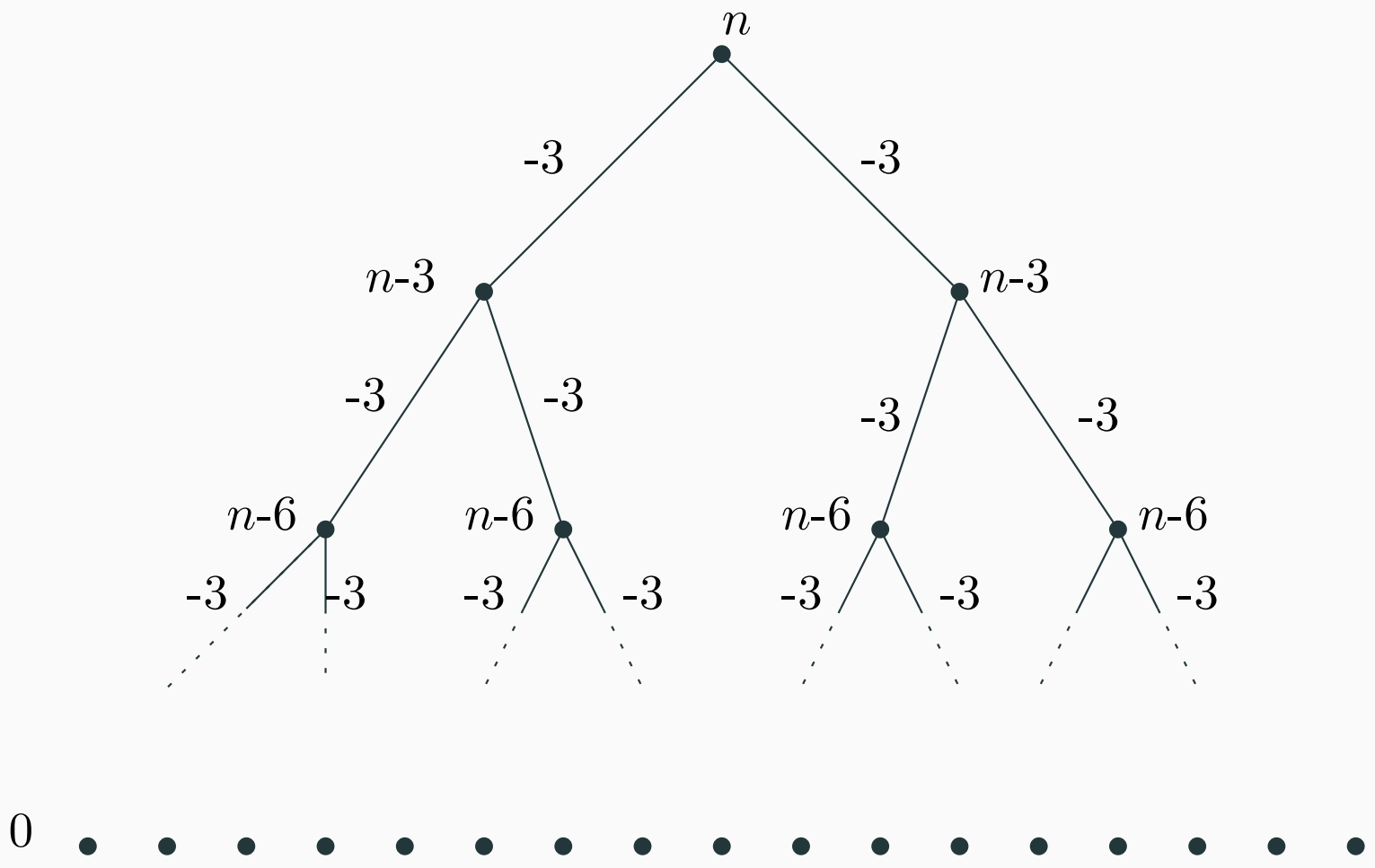

Figure 1: Tree with branching vector $(3, 3)$ at each node

example of tree with branching vector $(3, 3)$

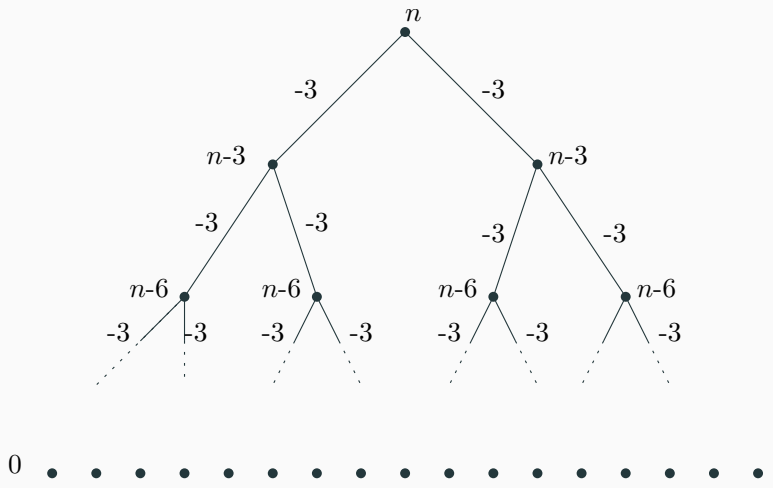

Figure 2: Tree with branching vector $(3, 3)$ at each node

How many leaves has this tree?

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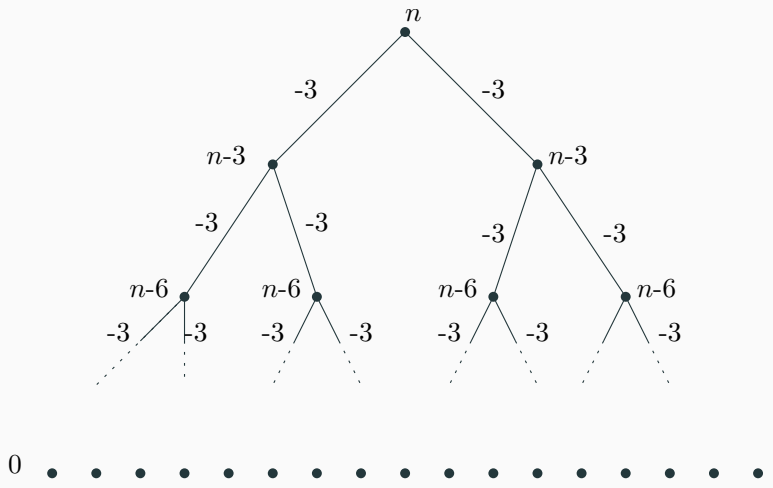

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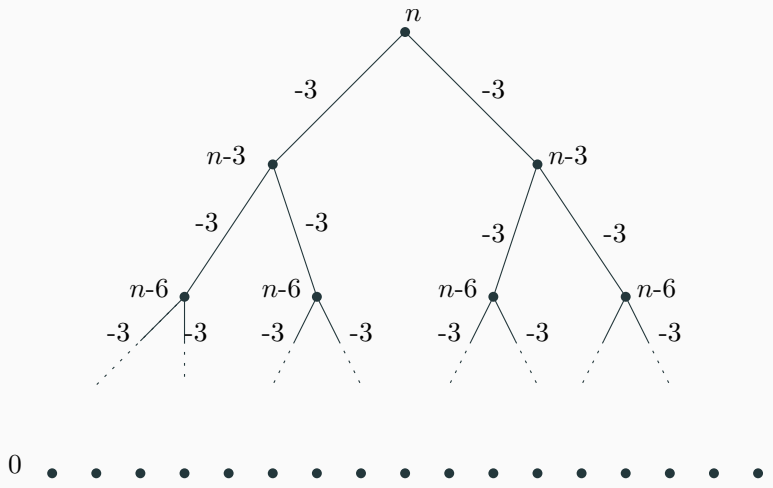

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Satisfies the recurrence $T(n) = T(n-3) + T(n-3)$

$T(n) = 2^{n/3} \sim 1.259^n$

Question: What is the number of leaves of a tree with a given branching vector $(x_1, x_2, ..., x_k)$

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Theorem

A tree with branching vector $(x_1, x_2, ..., x_k)$ has α^n leaves where α is the unique real positive solution of the equation:

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α is called the branching factor of the branching vector $(x_1, x_2, ..., x_k)$

back to our toy example

First Branching Rule:

- If F contains a degree 1 vertex u with neighbour v, then branch in the following ways:
  - Add u to S and remove u and v from F. $|F|$ decreases by 2
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Branching vector: $(2, 2)$

Branching factor of $(2, 2)$: $\sqrt{2} \sim 1.414$

back to our toy example

Second Branching Rule:

- If F contains a degree 2 vertex u with neighbour v, w, then branch in the following ways:
  - Add u to S and remove u, v and w from F. $|F|$ decreases by 3
  - Add v to S and remove u, v and all neighbours of v from F. $|F|$ decreases by 3
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Branching vector: $(3, 3, 3)$

Branching factor of $(3, 3, 3)$: $\sqrt[3]{3} \sim 1.442$

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To estimate the number of leaves $T(n)$ of the recursion tree (and then the complexity of the algorithm), we keep the worst branching factor

- Branching vector: $(2, 2)$ with branching factor $\sqrt{2} \sim 1.414$
- Branching vector: $(3, 3, 3)$ with branching factor $\sqrt[3]{3} \sim 1.442$

$T(n) \leq (\sqrt[3]{3})^n = 3^{n/3} \sim 1.442^n$

exercise

Problem: Exact 3-Sat enumeration

Input: A 3-CNF formula with m clauses and n variables

Output: All satisfying assignments of variables with exactly one true literal per clause

exercise

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Try to find an algorithm to solve this problem.

output sensitive enumeration

Total-Polynomial algorithm

output sensitive enumeration

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Definition

An enumeration algorithm is said to be total-polynomial (or output-polynomial) if its running time is

$$poly(m+n)$$

where $poly$ is a polynomial, n is the size of the input and m is the size of the output.

TotalP

Definition

We call TotalP the class of enumeration problems that admit a total-polynomial algorithm.

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How to prove that a problem is not in TotalP?

decision enumeration

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DecisionEnumeration problem

input: A set of solutions $\mathcal{S} := \{s_1, ..., s_k\}$

Question: Is $\mathcal{S}$ the set of all solution?

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Theorem

Let A be enumeration problem. If the DecisionEnumeration problem associated to A is NP-hard, then $P \notin \text{TotalP}$ unless $P = NP$.

proof

- Assume that there exists a polynomial P such that A can be solved in $P(n+m)$. And let $\mathcal{S} := \{s_1, ..., s_k\}$ be a set of solutions.

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- If the algorithm stops during this period, answer "Yes" to the question, all solutions are in $\mathcal{S}$.

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- Otherwise, stop it and answer "No".

decision enumeration

Theorem

Let A be enumeration problem. If the DecisionEnumeration problem associated to A is NP-hard, then $P \notin \text{TotalP}$ unless $P = NP$.

By abuse of terminology, we say sometimes that an enumeration problem is NP-hard when its associated DecisionEnumeration problem is so.

tractability?

Is the TotalP notion a good definition of tractable enumeration problems?

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Issue:
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They may all be given at the end of the algorithm.

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Can we refine the class TotalP to solve this issue ?

Incremental-Polynomial algorithm

incremental algorithm

An incremental-polynomial algorithm is a special case of total-polynomial algorithm.

Definition

An enumeration algorithm is said to be incremental-polynomial if the delay between the output of the k^{th} solution and the $(k+1)^{\text{th}}$ solution is bounded by $poly(n+k)$ where $poly$ is a polynomial and n is the input size.

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We call IncP the class of enumeration problems that admit a incremental-polynomial algorithm.

Proposition

$\textsc{IncP} \subseteq \textsc{TotalP}$

proof of the proposition

- If A is an incremental-polynomial algorithm, there exists a polynomial P such that the delay between the output of the k^{th} solution and the $(k+1)^{\text{th}}$ solution is bounded by $P(n+k)$
- If m is the total number of solutions, the total running time of the algorithm is

$$\sum_{i=1}^{m} P(n+i) \leq mP(n+m)$$

  which is polynomial in $n+m$

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The classes $\textsc{IncP}$ and $\textsc{TotalP}$ are well separated

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Proposition

Unless $P = NP$, $\textsc{IncP} \subsetneq \textsc{TotalP}$ ($\textsc{IncP} \neq \textsc{TotalP}$).

NewSolution problem

A classical way to obtain an incremental-polynomial algorithm is to be able to solve a slightly different version of the DecisionEnumeration problem.

NEWSOLUTION problem

A classical way to obtain an incremental-polynomial algorithm is to be able to solve a slightly different version of the DECISIONENUMERATION problem.

NEWSOLUTION problem

input: A set of solutions $\{s_1, ..., s_k\}$

output: "No" if $\{s_1, ..., s_k\}$ contains all solutions

A new solution $s \notin \{s_1, ..., s_k\}$ otherwise

new solution problem

To obtain a incremental-polynomial algorithm, one can apply successively the NewSolution problem by adding the new solution found at each step to the set of already generated solutions until the answer is "No".

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Proposition

Let A be an enumeration problem. Then $A \in \text{IncP}$ if and only if the NewSolution problem associated to A is polynomial.

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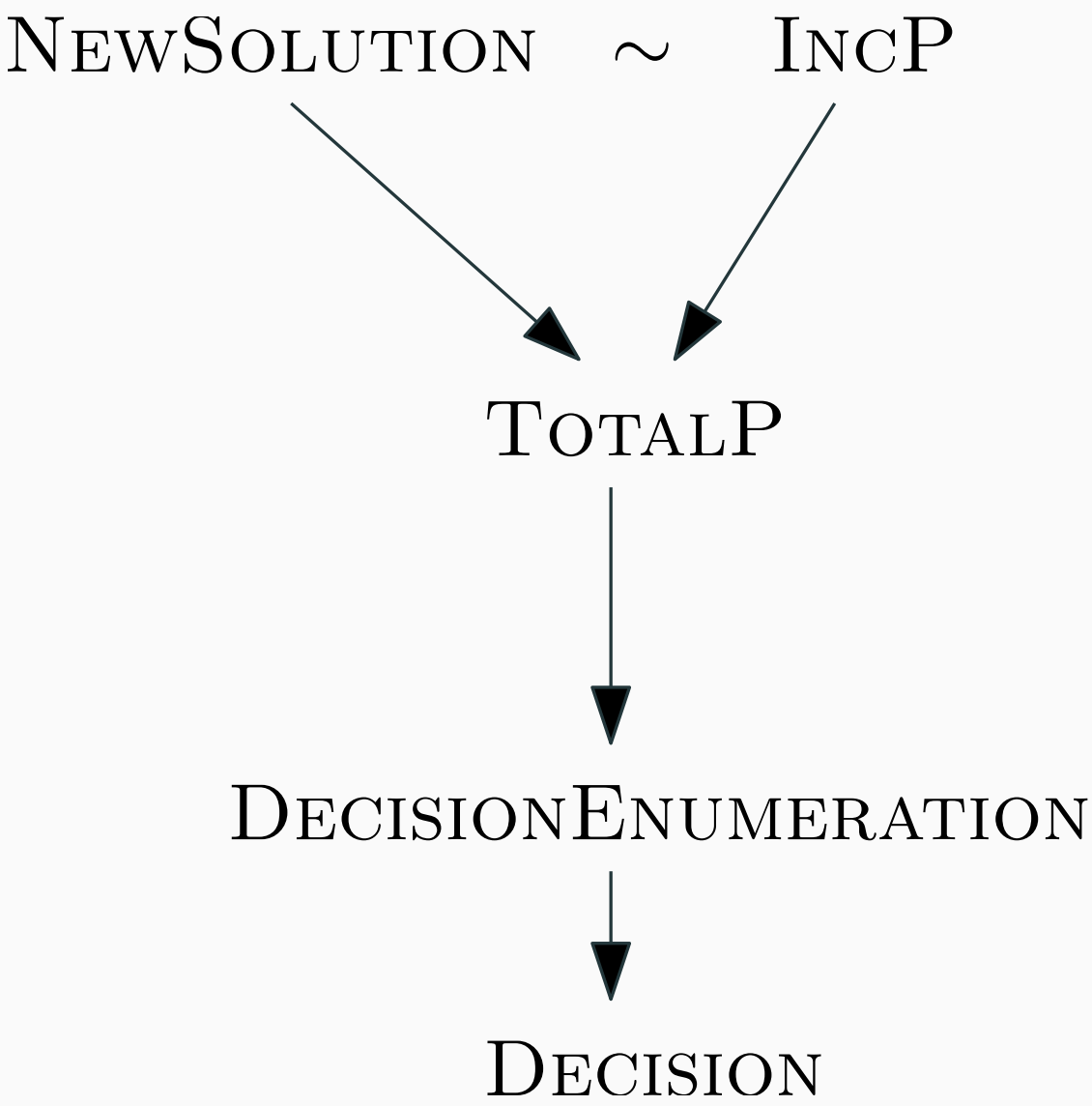
Let A be an enumeration problem. Then $A \in \text{IncP}$ if and only if the NewSolution problem associated to A is polynomial.

Proof: Left as exercise.

summarize



example: spanning trees of a graph

Let G be a graph and let T be a spanning tree of G. The fundamental cycle of T with respect to an edge $e \in E(G) \setminus E(T)$, is the unique simple cycle $F(T, e)$ of $T \cup \{e\}$.

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Proposition

A set $\mathcal{T}$ is the set of the spanning trees of a graph G, if and only if for all $T \in \mathcal{T}$ and for all edge $e \notin T$, the tree $(T \cup \{e\}) \setminus \{y\}$ belongs to $\mathcal{T}$ for all $y \in F(T, e)$.

homework: cycles of a graph

Let G be a graph, T one of its spanning tree and let $F(T)$ be the set of fundamental cycles of T.

Algorithm:

- output $F(T)$
- $S = F(T)$
- While $\exists\ C_1, C_2 \in S$, $C_1 \cap C_2 \neq \emptyset$ and $C_1 \Delta C_2 \notin S$:
  - output $C_1 \Delta C_2$
  - Add $C_1 \Delta C_2$ to S

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2. Prove or disprove it.

homework: cycles of a graph

Let G be a graph, T one of its spanning tree and let $F(T)$ be the set of fundamental cycles of T.

Algorithm:

- output $F(T)$
- $S = F(T)$
- While $\exists\ C_1, C_2 \in S$, $C_1 \cap C_2 \neq \emptyset$ and $C_1 \Delta C_2 \notin S$:
  - output $C_1 \Delta C_2$
  - Add $C_1 \Delta C_2$ to S

1. Does this algorithm enumerate all cycles of the graph?
2. Prove or disprove it.
3. Is it total-polynomial? Incremental-polynomial?

an open problem

- A hypergraph $\mathcal{H} := (V, \mathcal{E})$ is a set family $\mathcal{E}$ on a ground set V.

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Transversal DecisionEnumeration problem

input: A hypergraph $\mathcal{H}$ and a set of minimal transversals of $\mathcal{H}$ $\mathcal{S} := \{T_1, ..., T_k\}$

Question: Is $\mathcal{S}$ the set of all minimal transversals of $\mathcal{H}$?

transversal problem

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But we know that the NewSolution problem is quasi-polynomial (there is an algorithm of complexity N^{logN} where $N = k + |\mathcal{H}|$)

transversal DECISIONENUMERATION problem

Therefore :

- The minimal transversal of a hypergraph can be enumerated in incremental quasi-polynomial time
- Unless every NP-complete problem can be solved in quasi-polynomial time, the transversal DecisionEnumeration problem is not Np-complete.
- Open: Is the transversal (DecisionEnumeration/ NewSolution) problem polynomial?

example of problems in INCP

The enumeration of

- Spanning trees
- (Simple) Cycles
- (maximal) Paths, induced paths ...
- (maximal/maximum) matching
- maximal Independent sets
- maximal cliques/bicliques
- minimal vertex covers
- minimal feedback vertex/arc set
- ...

example np-hard enumeration problems

The enumeration of

- Minimal dicut of an oriented graph.
- Vertices of a polyhedron given by a set of inequalities.
- Maximal elements of an independent system given by an oracle.
- Any problem for which the (classical) decision problem is NP-hard.
- ...

equivalent to minimal transversals enumeration

The enumeration of

- Maximal stable sets of a hypergraph.
- All satisfying assignments of a monotone CNF.
- Minimal dominating sets of a graph.
- Minimal subsets that are not included in any set from a given family of sets.
- ...

Polynomial delay algorithm

issues

Issues of the Incremental algorithms:

- The delay between two consecutive solutions is increasing as more solutions are outputted.
- The space used by the algorithm is usually exponential in the input size.

polynomial delay algorithm

Definition

A polynomial delay algorithm is an enumeration algorithm such that the delay between two consecutive outputs is bounded by $poly(n)$ where $poly$ is a polynomial and n is the input size.

Supergraph method

supergraph method

A common way to obtain polynomial algorithms is the so called supergraph method.

Idea:

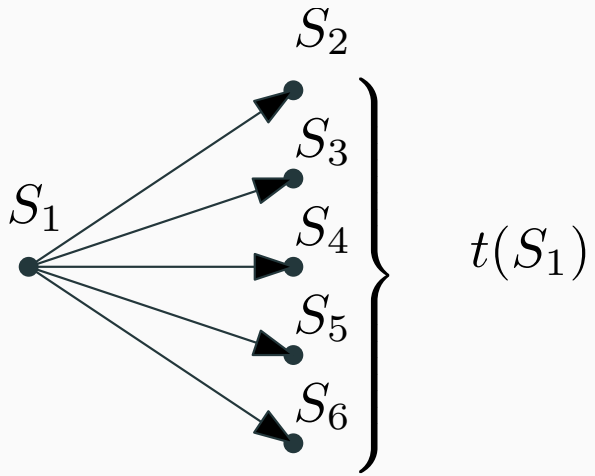
- Make small modifications on solutions to obtain new ones.
- Prove that all solutions can be obtained in this way starting from a special solution (or set of solutions).
- Try to avoid redundancy (output several times the same solution) using only polynomial space.

supergraph method

Let $\mathcal{S}$ be the set of solutions of a given enumeration problem.

The supergraph method consists on defining a transition function $t : \mathcal{S} \to 2^{\mathcal{S}}$.

It defines the arcs of an oriented graph with vertex set $\mathcal{S}$



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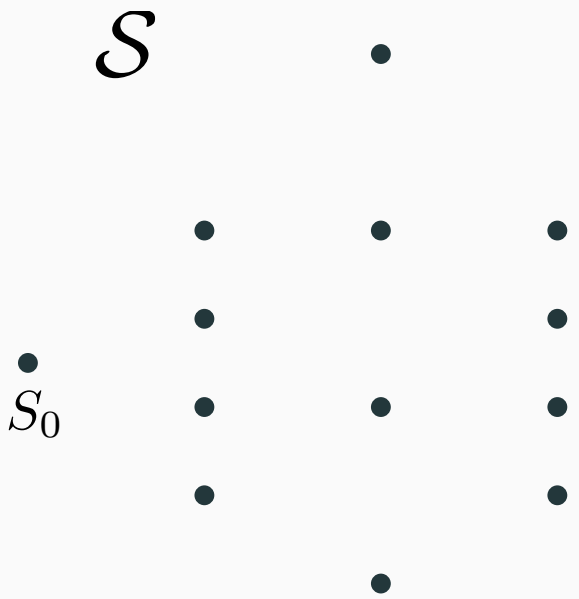

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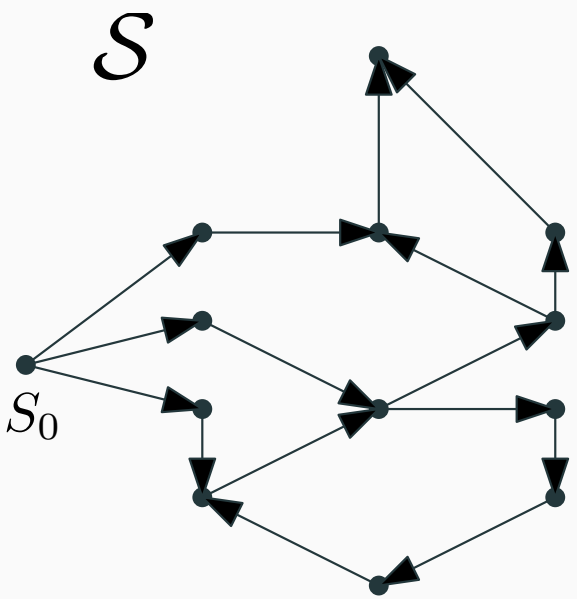
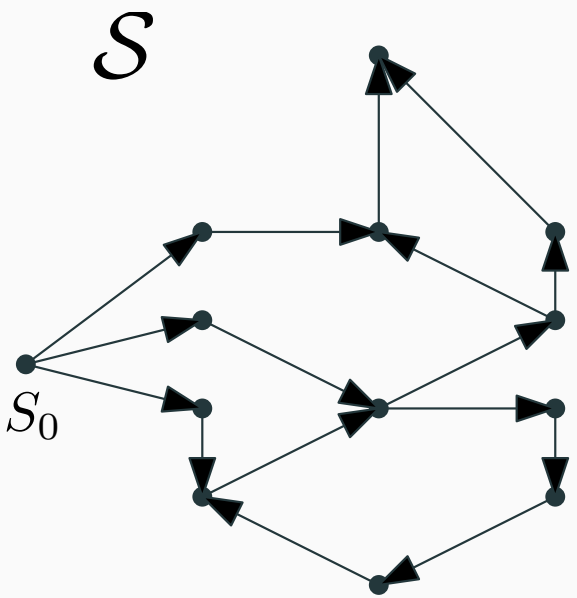


supergraph method

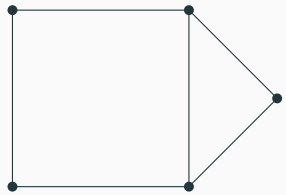
Assuming that the transition function is such that every solution is accessible from the starting solution, we would like to explore the graph (for instance with a DFS or a BFS) to find all solutions.

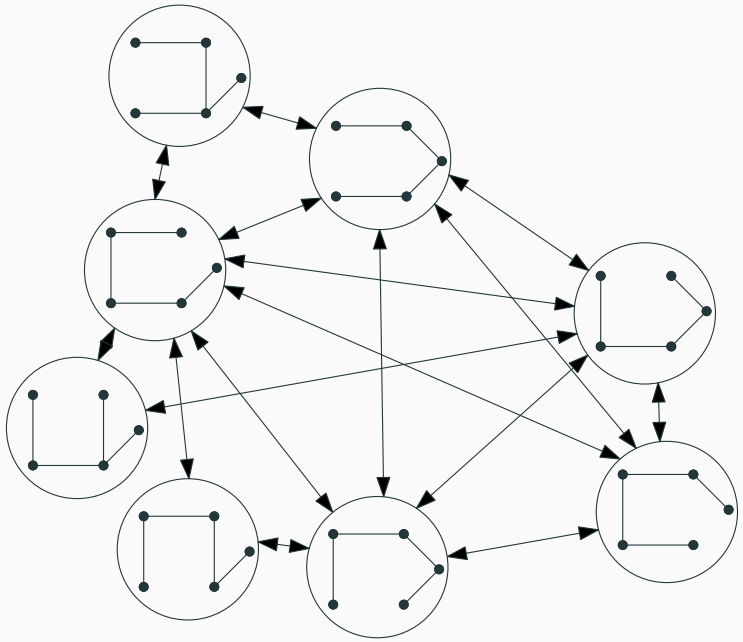


exemple: spanning trees

For a spanning tree T of a graph G, we define the following transition function:

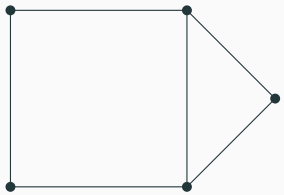
$$t(T) := \{(T \cup \{e\}) \setminus \{e'\} : e \notin T,\ e' \in F(T, e)\}$$

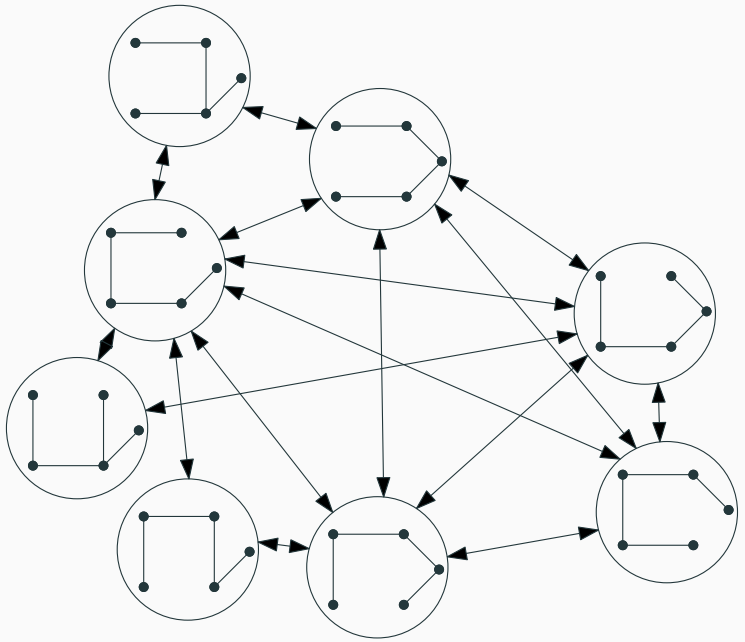




Issues:

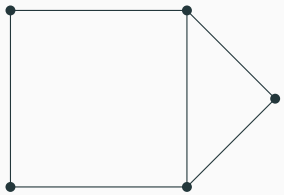
- When you explore a graph, you have to store the set of already visited vertices.
- Here the number of vertices of a supergraph corresponds to the number of solutions which is huge in general.

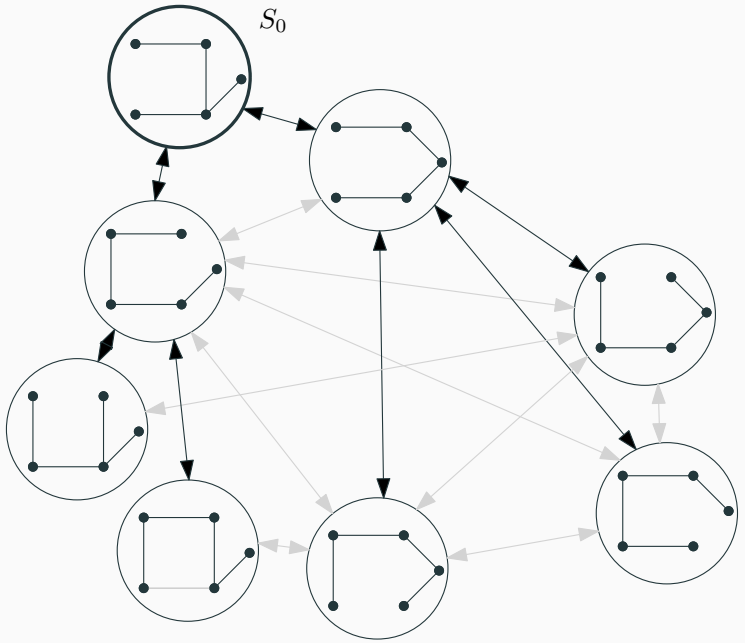




Solution:

- Restrict the transition function so that it defines an arborescence.
- The goal is to define a parent-child relation between solutions so that each solution will be "produced" only by its father





reverse search

To define an arborescence, we usually use the reverse search method.

- Define a total ordering over solutions.
- The father $P(S)$ of a solution S is defined as the smallest solution S' such that $S \in t(S')$

Then we restrict the transition function t to t' :

$$t'(T) := \{T' \in t(T) : P(T') = T\}$$

spanning trees

- Consider an ordering over the edges of G : $(e_1, ..., e_k)$.
- The ordering over spanning trees is given by the lexicographic order.
- The father $P(T)$ of a spanning tree T is the spanning tree $(T \cup \{e\}) \setminus \{e'\}$ where e is smallest edge that does not belong to T and e' is the largest edge of $F(T, e)$

algorithm

We call the algorithm with the starting solution S_0.

ReverseSearch(S):

- Output S
- For all $S' \in t(S)$
  - If $P(S') = S$
    - ReverseSearch(S')

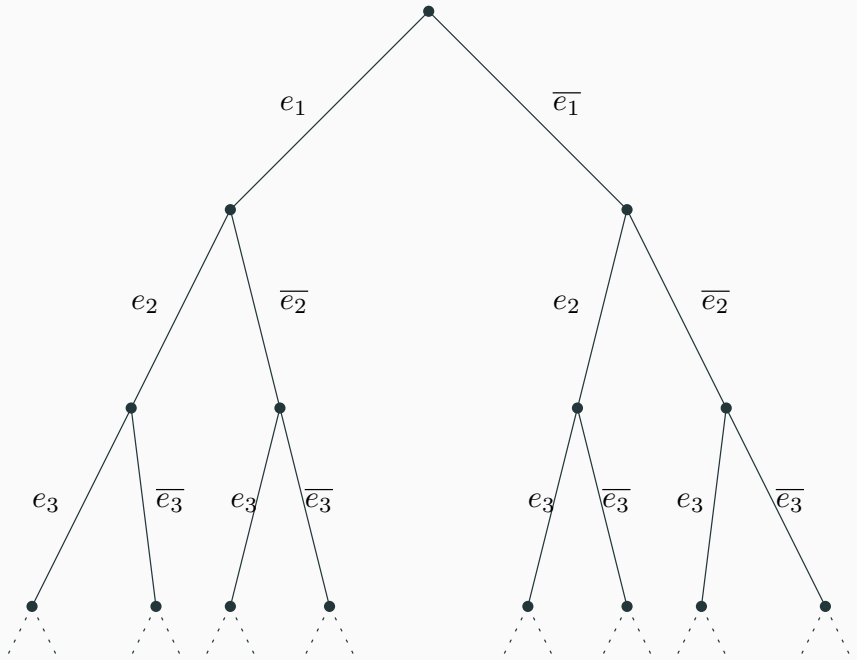
Backtracking method

flashlight

Here we assume that the solutions of the problem are subsets of a ground set E.

Steps

- The solutions corresponds to leaves of the tree.



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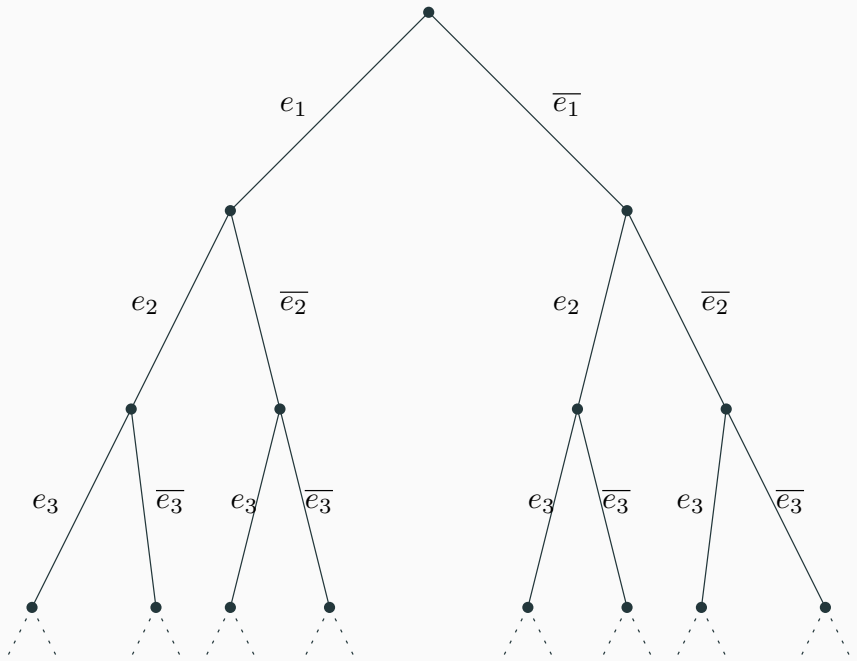
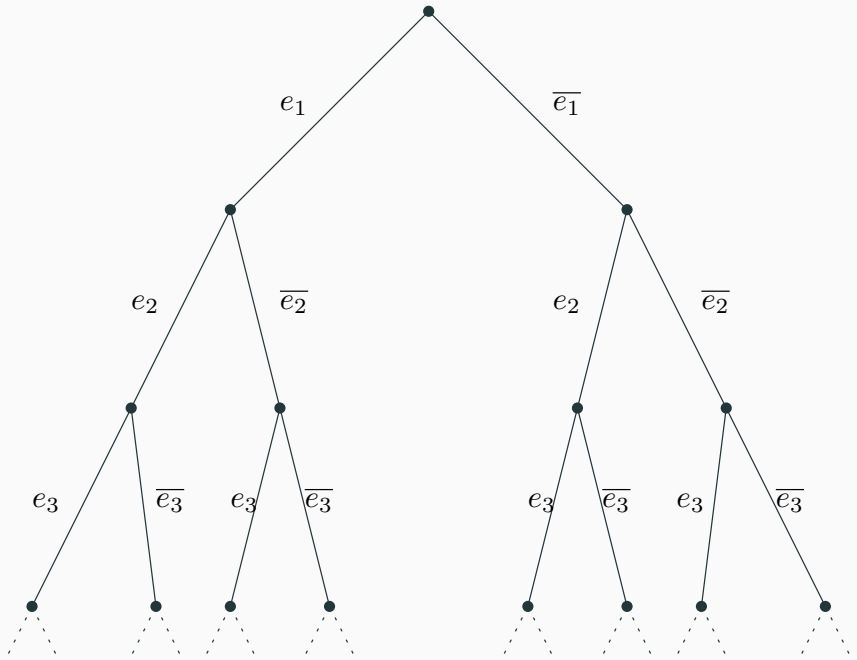


flashlight

Here we assume that the solutions of the problem are subsets of a ground set E.

Steps

- The solutions corresponds to leaves of the tree.
- Each level i corresponds to the choice of the selection of e_i in the solution.
- All the solutions in the left tree contain e_i and the ones of the right tree do not contain e_i.



extension problem

To use the backtracking method, we need to be able to solve the following problem.

Extension problem

Input: Two sets $X, Y \subseteq E$

Question: Is there a solution S such that $X \subseteq S$ and $Y \cap S = \emptyset$?

If the extension problem is polynomial, then the enumeration problem can be solved with polynomial delay.

algorithm

$\textsc{BacktrackAlgorithm}(i, X, Y)$:

- if $i = n$:
  - output X
- if there is a solution S such that $(X \cup \{e_i\}) \subseteq S$ and $Y \cap S = \emptyset$:
  - $\textsc{BacktrackAlgorithm}(i + 1, X \cup \{e_i\}, Y)$
- if there is a solution S such that $X \subseteq S$ and $(Y \cup \{e_i\}) \cap S = \emptyset$:
  - $\textsc{BacktrackAlgorithm}(i + 1, X, Y \cup \{e_i\})$

exercise

- Why the previous algorithm enumerates all solutions with polynomial delay (Assuming that the extension problem is polynomial)?
- Is the Extension problem is polynomial for spanning trees?
- Is it polynomial for the cycles of a graph?